Geometric Unity III: The Quantum Legality Layer

Semidirect BRST/BV, Anomaly Closure, Counterterm Classification, Quantum Projection–Variation, and the Axial Sign Corridor

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Abstract

Paper III is the quantum legality certificate for the Geometric Unity series. GU I supplies the completed classical geometry; GU II supplies the anomaly-safe matter and current ledger; GU III certifies which of those upstream objects can be gauge-fixed, quantized, renormalized, matched, and handed to GU IV without hidden anomalies, sign drift, or boundary obstructions. The result is a precise legality packet for the observable layer: a nilpotent semidirect BRST complex for the affine gauge bookkeeping, a BV master functional satisfying the classical master equation under declared closure hypotheses, a parity-even local cohomology and counterterm ledger through the dimension-six slice EFT order, anomaly closure in the quantized GU II field basis, a sign-corridor status ledger for the axial contact, a sterile dense-fermion import-legality lane, and a quantum Projection–Variation identity under admissible BRST-preserving boundary data.

The completed gauge variables are fixed by Paper I:

$$\widehat{A} = A - B, \qquad \widehat{F} = F(\widehat{A}).$$

For the semidirect ghosts c and η, the BRST differential is

$$sA = -D_A c - \eta, \qquad sB = -[B, c] - \eta, \qquad sc = -\frac{1}{2}[c, c], \qquad s\eta = -[\eta, c], \qquad s^2 = 0.$$

The completed variables therefore obey

$$s\widehat{A} = -D_{\widehat{A}} c, \qquad s\widehat{F} = -[\widehat{F}, c].$$

The affine ghost cancels from the completed connection, which is the first visible legality check for the completed-variable formulation.

The axial contact is matched only in the sign-safe operator basis

$$\Delta L_X = C_{55}(\mu) O_{55}, \qquad O_{55} = -J_{5\mu} J_5^{\mu}.$$

The sign $C_{55} > 0$ is inherited from the auxiliary-torsion matching of Paper I. Its propagation to the GU IV effective scale is exported either as a theorem when an anomalous-dimension matrix and invariant positive cone are supplied, or as a declared sign-corridor assumption when those data are outside the theorem domain. Forward-limit positivity is treated as a diagnostic, not as an automatic proof of every dimension-six four-fermion sign.

GU III also records the import conditions for the sterile dense-fermion branch defined in GU II. That branch enters GU III as source-class data requiring anomaly, washout, operator-basis, BRST/BV, boundary, and sign-corridor legality before GU IV may consume it through a declared source-adapter packet. BRST legality, anomaly closure, and sign-corridor status certify import consistency; they do not by themselves imply cosmological viability. GU III introduces no new matter, no new geometry, and no new observables. It closes the quantum legality layer and hands GU IV a certified packet from which observables may be built.

Keywords: Geometric Unity; BRST; BV formalism; semidirect gauge symmetry; affine shift; completed variables; local BRST cohomology; anomaly cancellation; counterterms; RG matching; axial contact; axial sign corridor; sterile dense-fermion branch; positivity diagnostics; quantum Projection–Variation; boundary BRST.

Executive Summary

Paper III establishes the quantum legality layer of the Geometric Unity series. It determines whether the completed fields of GU I and the matter/current ledger of GU II define a gauge-consistent, anomaly-free, counterterm-controlled, boundary-admissible EFT packet for GU IV. The layer preserves the GU I–II upstream packet and certifies the conditions under which GU IV may use it.

The current series records are:

GU I: 10.5281/*zenodo*.17252988, GU II: 10.5281/*zenodo*.17254875,

GU III: 10.5281/*zenodo*.17374258, GU IV: 10.5281/*zenodo*.17374850,

GU V: 10.5281/*zenodo*.17402260.

The key completed gauge objects are

$$\widehat{A} = A - B, \qquad \widehat{F} = F(\widehat{A}). \tag{0.1}$$

The semidirect BRST differential is

$$sA = -D_A c - \eta, \qquad sB = -[B, c] - \eta, \tag{0.2}$$

$$sc = -\frac{1}{2}[c, c], \qquad s\eta = -[\eta, c]. \tag{0.3}$$

The central cancellation is

$$s\widehat{A} = s(A - B) = -D_{\widehat{A}} c, \qquad s\widehat{F} = -[\widehat{F}, c]. \tag{0.4}$$

This is the simplest visible test that the affine ghost has disappeared from the completed variable.

Central scope sentence.

GU III preserves the GU I–II upstream packet and certifies its quantum-legality status for GU IV. It adds no new matter, no new geometry, and no new observable map.

Legality and viability. BRST legality, anomaly closure, and sign-corridor status certify import consistency. They do not by themselves imply cosmological viability. GU IV supplies the observable-interface tests, and GU V audits declared packets.

Principal theorem chain.

G1. Read-only upstream imports. GU III consumes the GU I completed variables and the GU II matter/current ledger without redefining $\widehat{A}$, $\widehat{F}$, T_{aug}, O_{55}, J_5^μ, $W_{\mathrm{fam}}^{\mathrm{LH}}$, or the hypercharge map.

G2. Semidirect BRST complex. The affine gauge pair and ghosts carry a nilpotent BRST differential for the field-level semidirect symmetry $H \ltimes N$.

G3. Completed-variable covariance. The affine ghost cancels in $\widehat{A} = A - B$, giving $s\widehat{A} = -D_{\widehat{A}}c$ and $s\widehat{F} = -[\widehat{F}, c]$.

G4. Completed torsion import. The torsion block is BRST-covariant when the GU I completed-spin cocycle is imported in the canonical or explicitly equivariant sector.

G5. BV closure. The minimal BV functional satisfies the classical master equation under the declared off-shell closure hypotheses.

G6. Counterterm ledger. The parity-even local cohomology through dimension six is represented by the declared slice EFT counterterm basis, including O_{55}.

G7. Anomaly closure. The GU II anomaly-safe matter table remains anomaly-free after the ghost, boundary, and measure ledgers are included under admissible boundary conditions.

G8. Axial sign corridor. The sign-safe axial contact $C_{55}(\mu)O_{55}$ is legal at the GU IV effective scale only under a cone-invariance theorem or a declared sign-corridor assumption.

G9. Sterile dense-fermion import lane. The sterile dense-fermion branch is imported only as GU II source-class data and requires GU III anomaly, washout, energy-budget, BRST/BV, boundary, operator-basis, and sign-corridor legality before downstream use.

G10. Quantum Projection–Variation. Gauge-fixed/BV pullback-variation compatibility holds modulo BRST-exact and admissible boundary terms.

Table 1: GU III export contract.

Export	Content	Consumer
GU3-A	Semidirect BRST complex for $H \ltimes N$, explicit gradings, and proof of $s^2 = 0$.	GU IV, GU V audit

Export	Content	Consumer
GU3-B	Gauge fixing, nonminimal sector, BV master action, and $(S_{\mathrm{BV}}, S_{\mathrm{BV}}) = 0$ under closure hypotheses.	GU IV boundary/EFT layer
GU3-C	Local cohomology/counterterm ledger through parity-even dim≤ 6, including O_{55}.	GU IV slice EFT
GU3-D	Anomaly closure by GU II import plus ghost, boundary, and measure checks.	GU IV legality
GU3-E	RG/matching map for $C_{55}(\mu)$ and σ_0, with sign-corridor status declared.	GU IV observable packet
GU3-F	Quantum Projection–Variation and boundary BRST admissibility.	GU IV, GU V
GU3-G	Positivity/causality diagnostics with declared amplitude and subtraction scope.	GU IV, GU V
GU3-H	Sterile dense-fermion branch import-legality conditions: anomaly, washout, energy budget, BRST/BV, boundary, operator basis, and sign-corridor gates.	GU IV source adapters; GU V audit

Read-only upstream packet. GU III preserves the completed variables of GU I, the matter representation table of GU II, and the observable definitions assigned to GU IV. The read-only upstream objects are

$$\widehat{A}, \qquad \widehat{F}, \qquad T_{\mathrm{aug}}, \qquad O_{55}, \qquad J_5^{\mu}, \qquad W_{\mathrm{fam}}^{\mathrm{LH}}, \qquad Y = T_R^3 + \frac{1}{2}(B - L).$$

A change to any of these objects opens a separate compatibility ledger before GU IV may consume the modified packet.

Standard-Physics Dictionary

Table 2: GU III terminology translated into standard mathematical-physics language.

GU III term	Standard meaning
Quantum legality layer	BRST/BV, anomaly, counterterm, RG/matching, and boundary-consistency layer between classical geometry/matter and observables.
Semidirect BRST complex	BRST differential for the field-level affine semidirect symmetry $H \ltimes N$, with H-ghost c and affine ghost η.
Affine ghost	Ghost for the one-form affine shift sector $N \simeq \Omega^1(Y, \operatorname{ad} P)$.
Completed variable	Affine-invariant or affine-covariant combination, especially $\widehat{A} = A - B$.
Local cohomology ledger	Classification of counterterms in local BRST cohomology, here represented by $H^{0,4}(s\|d)$ on the slice.
Axial sign corridor	RG/matching domain where $C_{55}(\mu)$ remains in the positive O_{55} cone between the GU matching scale and the GU IV effective scale.
Positive O_{55} cone	Operator-basis statement $C_{55} > 0$ for $O_{55} = -J_5^2$, not a statement about the raw mostly-plus contraction J_5^2.
Positivity diagnostic	Amplitude/dispersion check that can test consistency but does not automatically prove every dimension-six four-fermion sign.
Quantum Projection–Variation	Gauge-fixed/BV version of pullback-variation compatibility modulo BRST-exact and admissible boundary terms.
Sterile dense-fermion import legality	GU III legality lane for the GU II sterile dense-fermion source-class packet, including anomaly, washout, BRST/BV, boundary, operator-basis, and sign-corridor gates.

Main Theorem

Theorem 0.1 (GU III Quantum Legality Export Theorem)**.** *Assume the GU I completed-variable convention ledger, the GU II anomaly-safe matter/current ledger, a declared field-level affine semidirect algebra $H \ltimes N$, the grading conventions of Section A, admissible BRST-preserving boundary families, and the parity-even dim≤ 6 slice EFT operator basis declared in Section E. Then GU III proves the following legality exports.*

(i) *The semidirect BRST differential*

$$sA = -D_A c - \eta, \qquad sB = -[B, c] - \eta, \tag{0.5}$$

$$sc = -\frac{1}{2}[c, c], \qquad s\eta = -[\eta, c] \tag{0.6}$$

is nilpotent on the affine gauge pair and ghosts:

$$s^2 A = s^2 B = s^2 c = s^2 \eta = 0. \tag{0.7}$$

(ii) *The completed gauge variables obey*

$$\widehat{A} = A - B, \qquad s\widehat{A} = -D_{\widehat{A}} c, \qquad s\widehat{F} = -[\widehat{F}, c]. \tag{0.8}$$

(iii) *The completed torsion block is BRST-covariant under the GU I completed-spin import condition:*

$$s\widehat{\omega}_B = -D_{\widehat{\omega}_B} \lambda, \qquad sT_{\text{aug}} = -[\lambda, T_{\text{aug}}]_{\text{Lor}}, \tag{0.9}$$

where λ is the Lorentz/spin ghost and the bracket is the relevant representation action.

(iv) *The minimal BV functional*

$$S_{\text{BV}} = S_{\text{cl}} + \int \langle A^*, sA \rangle + \int \langle B^*, sB \rangle + \int \langle c^*, sc \rangle + \int \langle \eta^*, s\eta \rangle + \cdots \tag{0.10}$$

satisfies the classical master equation under the declared off-shell closure hypotheses:

$$(S_{\text{BV}}, S_{\text{BV}}) = 0. \tag{0.11}$$

(v) *The parity-even local BRST cohomology through the GU IV slice order is represented by the counterterm ledger*

$$H^{0,4}(s|d)_{\dim \leq 6, \text{even}} = \text{span}\{\textit{physical classes in Table 8}\} \oplus s(\cdots) \oplus d(\cdots) \oplus \text{EOM}. \tag{0.12}$$

(vi) *The total anomaly ledger for the quantized GU III field basis is*

$$\mathcal{A}_{\text{GU III}} = \mathcal{A}_{\text{GU II}} + \mathcal{A}_{\text{ghost}} + \mathcal{A}_{\text{boundary}} + \mathcal{A}_{\text{measure}} = 0, \tag{0.13}$$

under the import and boundary conditions of Section F.

(vii) *The axial contact is matched in the sign-safe basis*

$$\Delta L_X = C_{55}(\mu) O_{55}, \qquad O_{55} = -J_{5\mu} J_5^{\mu}. \tag{0.14}$$

The sign at the matching scale is inherited from the GU I auxiliary torsion theorem. Propagation of $C_{55}(\mu) > 0$ to the GU IV effective scale is legal either by the cone-invariance theorem of theorem 8.2 or by a declared sign-corridor assumption.

(viii) *Quantum Projection–Variation holds for BRST-closed physical variations modulo BRST-exact and admissible boundary terms:*

$$\delta\Gamma_X[\iota^*\Phi] - \iota^*(\delta\Gamma_Y[\Phi]) = s\mathcal{K}_X + d_{\partial X}\mathcal{B}_X, \tag{0.15}$$

with $\int_{\partial X} \mathcal{B}_X = 0$ for the declared admissible boundary family.

(ix) *The sterile dense-fermion branch imported from GU II is legal for downstream consumption only when its source-class packet satisfies the GU III import conditions:*

$$\mathcal{A}_{\text{sterile}} = 0, \qquad \Gamma_\chi \tau_{\text{prod}} \lesssim 1, \qquad \rho_{\text{deg}} \lesssim \rho_b, \qquad C_{55}(\mu_{\text{eff}}) > 0 \textit{ status recorded.} \tag{0.16}$$

This legality statement certifies branch import. It does not compute a parent-source value, a tensor amplitude, a detector corridor, or a GU IV observable map.

Consequently, GU IV may import

$$\sigma_0^2 = C_{55}(\mu_{\text{eff}}) n_{5,0}^2 \tag{0.17}$$

only under the GU II conserved or effectively conserved axial-current branch and under the GU III sign-corridor status recorded in Table 4.

Proof. The BRST nilpotency and completed-variable covariance are proved in Section B. The completed-spin import condition is stated in Section 3 and used in Section B. The BV master equation is proved in Section D. Gauge fixing and the nonminimal sector are constructed in Section C. The local cohomology ledger is Section E. Anomaly closure is proved in Section F. The axial sign corridor and positivity diagnostics are separated in Sections G and H. The quantum Projection–Variation identity is proved in Section I. The sterile dense-fermion import lane is stated in Section 7. The GU IV import restrictions are collected in Sections 10 and K. □

Figures and Claim Status

$$\boxed{\text{GU I completed fields}} + \boxed{\text{GU II matter/current ledger}}$$
$$\Downarrow$$
$$H \ltimes N \text{ BRST} \Longrightarrow s^2 = 0 \Longrightarrow S_{\mathrm{BV}} \Longrightarrow H(s|d)$$
$$\Longrightarrow \mathcal{A}_{\mathrm{total}} = 0 \Longrightarrow C_{55}(\mu) \Longrightarrow \boxed{\text{GU IV legality packet}}$$

Figure 1: GU III legality pipeline. Paper III certifies which GU I–II exports survive quantization, gauge fixing, local counterterms, anomaly checks, RG/matching, and boundary Projection–Variation.

$$\begin{aligned} A &\xrightarrow{s} -D_A c - \eta, \\ B &\xrightarrow{s} -[B, c] - \eta, \\ \text{subtract:} \quad \widehat{A} &= A - B \quad \Longrightarrow \quad s\widehat{A} = -D_{\widehat{A}} c. \end{aligned}$$

Figure 2: Completed-variable BRST cancellation. The affine ghost η cancels from the completed connection.

$$C_{55}(\mu_{\mathrm{GU}}) > 0 \quad \xrightarrow{\mu \frac{dC_i}{d\mu} = \frac{1}{16\pi^2}\gamma_{ij} C_j} \quad \mathcal{C}_{55}^{+} \quad \Longrightarrow \quad C_{55}(\mu_{\mathrm{eff}}) > 0$$
$$\sigma_0^2 = C_{55}(\mu_{\mathrm{eff}}) n_{5,0}^2.$$

Figure 3: Axial sign corridor. The sign corridor is a closed RG/matching domain, not a generic positivity slogan.

Table 3: Claim-status table for GU III.

Claim	**Status in GU III**
$s^2 = 0$ for A, B, c, η	Theorem; proved in Section B.
Completed curvature BRST covariance	Theorem; $s\widehat{A} = -D_{\widehat{A}} c$, $s\widehat{F} = -[\widehat{F}, c]$.
Completed torsion BRST covariance	Theorem under GU I completed-spin import condition.
BV master equation	Theorem under off-shell closure and imported spin-cocycle hypotheses.
Gauge fixing	Construction with nonminimal fields, gauge-fixing fermion, and ghost operators.

Claim	**Status in GU III**
Ghost kinetic/shift operators	Construction/check in Section C.
Local cohomology dim≤ 6	Ledger/table representing $H^{0,4}(s\|d)$ modulo exact and redundant classes.
Anomaly closure	Theorem by GU II import plus ghost, boundary, and measure checks.
RG sign corridor	Conditional theorem if γ_{ij} and invariant cone are supplied; otherwise declared sign-corridor assumption.
Positivity bounds	Diagnostic unless full amplitude, subtraction, and operator-dimension assumptions are supplied.
Quantum Projection–Variation	Theorem under admissible BRST-preserving boundary family.
Sterile dense-fermion branch legality	Import-legality lane; branch remains GU II source-class data until GU III anomaly, washout, BRST/BV, boundary, operator-basis, and sign-corridor gates are recorded.
GU IV import packet	Export table; GU IV may import only listed objects under stated conditions.

GU IV Export Ledger

Table 4: Objects GU IV may import from GU III.

Export	Import condition
GU3-A	$s^2 = 0$, $s\widehat{A} = -D_{\widehat{A}}c$, $s\widehat{F} = -[\widehat{F}, c]$, proved in the affine completed sector.
GU3-B	$(S_{\mathrm{BV}}, S_{\mathrm{BV}}) = 0$ under off-shell closure and completed-spin import hypotheses.
GU3-C	$H^{0,4}(s\|d)_{\dim\leq 6,\mathrm{even}}$ represented by the declared counterterm ledger.
GU3-D	$\mathcal{A}_{\mathrm{total}} = 0$ by GU II anomaly import plus ghost, boundary, and measure checks.
GU3-E	$C_{55}(\mu_{\mathrm{eff}}) > 0$ by the cone-invariance theorem or by a declared sign-corridor assumption.
GU3-F	$\delta\Gamma_X - \iota^*\delta\Gamma_Y = s\mathcal{K}_X + d_{\partial X}\mathcal{B}_X$ under admissible BRST boundary data.
GU3-G	$\sigma_0^2 = C_{55}(\mu_{\mathrm{eff}})n_{5,0}^2$, legal only under the GU II conserved/effectively conserved axial branch and recorded sign-corridor status.
GU3-H	Sterile dense-fermion branch source-class packet, legal only after anomaly, washout, BRST/BV, boundary, operator-basis, energy-budget, and sign-corridor gates are recorded.

Reproducibility Protocol

A referee-facing GU III release is not only a theorem ledger; it is an import protocol. Downstream use of the legality packet should identify the checks, ledgers, and manifests used to support the BRST/BV, cohomology, anomaly, RG/matching, and boundary claims. The reproducibility protocol requires manifest entries with filename, version, date, checksum, and proof label for the following artifact classes whenever they are released:

Artifact class	**Purpose**
`brst_nilpotency_check`	verifies $s^2 = 0$ and completed-variable covariance.
`bv_master_check`	verifies the classical master equation under the declared closure hypotheses.
`cohomology_ledger`	records the dim≤ 6 local cohomology/counterterm basis.
`operator_basis_dim6`	fixes the four-fermion and axial operator basis containing O_{55}.
`anomaly_import_manifest`	records the GU II anomaly ledger imported by GU III.
`rg_matrix_O55` or `sign_corridor_assumption_manifest`	records either the supplied RG matrix/cone proof or the declared sign-corridor assumption.
`quantum_pv_boundary_ledger`	records the admissible BRST boundary family used for quantum Projection–Variation.
`sterile_dense_fermion_import_manifest`	records the branch-carrier, anomaly, washout, energy-budget, operator-basis, boundary, and sign-corridor gates for any sterile dense-fermion source-class import.
`gu3_export_manifest`	lists the objects GU IV may import and their proof labels.

If an artifact is not present in the manifest, the manuscript may state that the protocol requires it, but may not claim that the artifact has been released.

Contents

1 Inputs from GU I and GU II

Paper III imports only locked upstream objects. It does not redefine the Paper I completed geometry or the Paper II matter/current table.

Table 5: Import locks for GU III.

Imported object	**Source**	**GU III status**
$\widehat{A} = A - B$	GU I	Fixed completed connection.
$\widehat{F} = F(\widehat{A})$	GU I	Fixed completed curvature.
$T_{\mathrm{aug}} = T(\widehat{\omega}_B)$	GU I	Fixed as a classical object; BRST covariance conditional on completed-spin import.
$O_{55} = -J_5^2$	GU I	Fixed sign-safe axial operator basis.
$W_{\mathrm{fam}}^{\mathrm{LH}}$	GU II	Fixed matter representation.
GU II anomaly table	GU II	Consumed, not recomputed unless matter changes.
J_5^μ, n_5	GU II	Consumed current definitions.
Conserved axial branch	GU II	Conditional branch for $n_5(a) = n_{5,0}a^{-3}$.
Sterile dense-fermion source-class packet	GU II	Source-class branch requiring GU III anomaly, washout, energy-budget, boundary, BRST/BV, operator-basis, and sign-corridor legality before downstream use.
$\sigma_0^2 = C_{55}n_{5,0}^2$	GU II/III/GU IV bridge	Legal only after RG/matching status is declared.

Condition 1.1 (No-redefinition import condition)**.** GU III may introduce renormalized coefficients, schemes, antifields, ghosts, gauge-fixing choices, and counterterm coordinates. It may not redefine O_{55}, J_5^μ, $\widehat{A}$, $\widehat{F}$, T_{aug}, the GU II family representation, the GU II hypercharge map, or the official sterile dense-fermion branch name without opening a separate compatibility analysis.

The upstream axial operator is

$$O_{55} = -J_{5\mu}J_5^\mu. \tag{1.1}$$

The renormalized axial contact is therefore written

$$\Delta L_X = C_{55}(\mu)O_{55}. \tag{1.2}$$

The coefficient of the raw mostly-plus contraction $J_{5\mu}J_5^\mu$ is $-C_{55}(\mu)$. This sign translation is never suppressed in GU III.

2 Semidirect BRST Complex

Let c be the H-ghost and η the affine-shift ghost. The affine sector is treated as an abelian one-form module over the homogeneous gauge algebra. The BRST operator has ghost number $+1$, is odd, and acts as a graded derivation with respect to the total degree defined in Section A.

Definition 2.1 (Semidirect BRST differential)**.** The BRST differential on the affine gauge pair and ghosts is

$$sA = -D_A c - \eta, \tag{2.1}$$

$$sB = -[B, c] - \eta, \tag{2.2}$$

$$sc = -\frac{1}{2}[c, c], \tag{2.3}$$

$$s\eta = -[\eta, c]. \tag{2.4}$$

Here $D_A c = dc + [A, c]$, N is abelian, and the bracket is the graded bracket induced by the adjoint action of H on the affine sector.

Theorem 2.2 (Nilpotency of the semidirect BRST differential)**.** *With the gradings and bracket conventions of Section A, the differential (2.1)–(2.4) satisfies*

$$s^2 A = s^2 B = s^2 c = s^2 \eta = 0. \tag{2.5}$$

Proof. The explicit computation is given in Section B. The result follows from the graded Jacobi identity, the standard BRST closure of the homogeneous connection, the adjoint action of H on N, and the abelian nature of the affine shift sector. □

Proposition 2.3 (Completed connection covariance)**.** *For $\widehat{A} = A - B$, the semidirect BRST differential gives*

$$s\widehat{A} = -D_{\widehat{A}} c. \tag{2.6}$$

Consequently,

$$s\widehat{F} = -[\widehat{F}, c]. \tag{2.7}$$

Proof. Subtract (2.2) from (2.1):

$$\begin{aligned} s(A - B) &= -D_A c - \eta + [B, c] + \eta \\ &= -dc - [A, c] + [B, c] \\ &= -dc - [A - B, c] \\ &= -D_{\widehat{A}} c. \end{aligned} \tag{2.8}$$

This proves (2.6). Since $\widehat{F} = F(\widehat{A})$ is the curvature of the completed connection, the standard curvature covariance computation gives (2.7). □

3 Completed Spin/Torsion Import

BRST covariance of the completed torsion block depends on the completed spin-cocycle theorem imported from GU I. GU III makes this dependency explicit rather than burying it inside the gauge-sector BRST calculation.

Import Condition 3.1 (Completed spin BRST compatibility)**.** GU III imports from GU I the completed spin-cocycle theorem in the canonical torsion sector or in an explicitly equivariant noncanonical sector:

$$\widehat{\omega}_B^{(h,n)} = h^{-1}\widehat{\omega}_B h + h^{-1}dh. \tag{3.1}$$

The infinitesimal BRST form is

$$s\widehat{\omega}_B = -D_{\widehat{\omega}_B}\lambda, \qquad sT_{\mathrm{aug}} = -[\lambda, T_{\mathrm{aug}}]_{\mathrm{Lor}}, \tag{3.2}$$

where λ is the Lorentz/spin ghost and $[\cdot,\cdot]_{\mathrm{Lor}}$ denotes the induced Lorentz-representation action on torsion-valued forms.

Scope Note 3.2 (Gauge-sector versus torsion-sector legality)**.** The affine gauge-sector results $s^2 = 0$, $s\widehat{A} = -D_{\widehat{A}}c$, and $s\widehat{F} = -[\widehat{F}, c]$ do not require the completed spin import. The torsion-sector claims involving $\widehat{\omega}_B$ and T_{aug} use theorem 3.1. If that import condition is not imposed, GU III exports only the gauge-sector BRST legality and not the completed-torsion BRST legality.

4 BV Master Action and Gauge Fixing

Because the semidirect BRST algebra closes off shell in the affine gauge sector, the minimal BV action has the standard antifield-linear form. The completed spin/torsion terms are included under theorem 3.1. Gauge fixing is then implemented by a nonminimal sector and a gauge-fixing fermion.

Theorem 4.1 (Classical BV master equation)**.** *Let S_{cl} be a GU III classical action built from GU I completed covariant blocks and GU II anomaly-safe matter data. Assume the affine BRST transformations (2.1)–(2.4), the completed-spin import condition (3.2) for the torsion sector when included, and off-shell closure of the declared field algebra. Then the minimal BV functional*

$$S_{\mathrm{BV}} = S_{\mathrm{cl}} + \int \langle A^*, sA\rangle + \int \langle B^*, sB\rangle + \int \langle c^*, sc\rangle + \int \langle \eta^*, s\eta\rangle + \int \langle \Psi^*, s\Psi\rangle + \int \langle \omega_B^*, s\widehat{\omega}_B\rangle + \cdots \tag{4.1}$$

satisfies

$$(S_{\mathrm{BV}}, S_{\mathrm{BV}}) = 0. \tag{4.2}$$

No quadratic antifield terms are required in the closed off-shell sector.

Proof. The proof is given in Section D. The antibracket action $(S_{\mathrm{BV}}, \cdot)$ reproduces the BRST differential on all included fields. Since $sS_{\mathrm{cl}} = 0$ and $s^2 = 0$ under the declared closure hypotheses, the classical master equation follows. □

Construction 4.2 (Nonminimal sector and gauge-fixing fermion)**.** Introduce nonminimal pairs

$$s\bar{c} = b_c, \quad sb_c = 0, \qquad s\bar{\eta} = b_\eta, \quad sb_\eta = 0. \tag{4.3}$$

For the homogeneous gauge part one may use the Lorenz-type gauge

$$\mathcal{F}_A = \nabla^\mu A_\mu. \tag{4.4}$$

For the affine one-form shift, GU III uses a vector-valued shift gauge

$$\mathcal{F}_B^\mu = B^\mu \tag{4.5}$$

with optional supplementary derivative gauge when a longitudinal parametrization is declared. A compatible gauge-fixing fermion is

$$\Psi_{\mathrm{gf}} = \int_X \left[\langle \bar{c}, \mathcal{F}_A \rangle + \langle \bar{\eta}_\mu, \mathcal{F}_B^\mu \rangle + \frac{\xi_A}{2} \langle \bar{c}, b_c \rangle + \frac{\xi_B}{2} \langle \bar{\eta}_\mu, b_\eta^\mu \rangle \right]. \tag{4.6}$$

Antifields are eliminated by

$$\Phi^* = \frac{\delta \Psi_{\mathrm{gf}}}{\delta \Phi}. \tag{4.7}$$

The ghost and shift-ghost operators are displayed in Section C.

5 Local Cohomology, Counterterms, and the Axial Operator Basis

The local cohomology problem identifies which deformations survive gauge symmetry and which are exact, boundary, redundant, or removable by field redefinitions. GU III classifies the parity-even slice EFT through the order consumed by GU IV.

Definition 5.1 (Admissible counterterm)**.** An admissible counterterm is a local four-form density $\mathcal{O}$ on the slice such that

$$s\mathcal{O} + d\mathcal{K} = 0 \tag{5.1}$$

modulo equations of motion and field redefinitions, within the declared mass dimension and parity sector.

Export 5.2 (Dimensional cohomology export)**.** The counterterm classification exported to GU IV is the parity-even dim≤ 6 slice basis in Table 8, with the axial operator written as

$$O_{55} = -(\bar{\Psi}\gamma_\mu\gamma^5\Psi)(\bar{\Psi}\gamma^\mu\gamma^5\Psi). \tag{5.2}$$

The cohomology statement is modulo BRST-exact, total-derivative, equation-of-motion, and field-redefinition redundancies.

The four-fermion sub-basis containing the axial channel is

$$O_{AA} := -(\bar{\Psi}\gamma_\mu\gamma^5\Psi)(\bar{\Psi}\gamma^\mu\gamma^5\Psi) = O_{55}, \tag{5.3}$$
$$O_{VV} := (\bar{\Psi}\gamma_\mu\Psi)(\bar{\Psi}\gamma^\mu\Psi), \tag{5.4}$$
$$O_{VA} := (\bar{\Psi}\gamma_\mu\Psi)(\bar{\Psi}\gamma^\mu\gamma^5\Psi), \tag{5.5}$$
$$O_{LL} := (\bar{\Psi}_L\gamma_\mu\Psi_L)(\bar{\Psi}_L\gamma^\mu\Psi_L), \tag{5.6}$$
$$O_{RR} := (\bar{\Psi}_R\gamma_\mu\Psi_R)(\bar{\Psi}_R\gamma^\mu\Psi_R), \tag{5.7}$$
$$O_{LR} := (\bar{\Psi}_L\gamma_\mu\Psi_L)(\bar{\Psi}_R\gamma^\mu\Psi_R), \tag{5.8}$$

with internal indices contracted only in GU II-admissible singlet channels.

6 Anomaly Closure

GU III consumes the GU II anomaly ledger and checks that quantization, ghosts, measure choices, and admissible boundaries introduce no new anomaly obstruction in the exported field basis.

Theorem 6.1 (GU III anomaly closure)**.** *If GU III uses the matter content, chirality convention, and gauge group imported from GU II, then the total anomaly ledger is*

$$\mathcal{A}_{\text{GU III}} = \mathcal{A}_{\text{GU II}} + \mathcal{A}_{\text{ghost}} + \mathcal{A}_{\text{boundary}} + \mathcal{A}_{\text{measure}} = 0, \tag{6.1}$$

under the admissible boundary conditions of Section I.

Proof. The GU II matter anomaly ledger gives $\mathcal{A}_{\text{GU II}} = 0$. The homogeneous and affine ghosts are adjoint BRST gauge-sector fields and do not add new chiral matter triangle anomalies. Boundary parity densities are either exact or excluded by the admissible boundary family. The measure anomaly follows the same representation ledger as GU II. The detailed ledger is given in Section F. □

Condition 6.2 (Anomaly-consumption condition)**.** If GU III changes the matter content, chirality convention, gauge group, hypercharge embedding, or family table relative to GU II, then the GU II anomaly ledger is no longer imported and requires recomputation before any GU IV export is legal.

7 Sterile Dense-Fermion Branch Import Legality

GU II defines the sterile dense-fermion branch as a species-resolved weak-singlet source class with polarization, washout, and energy-budget gates. GU III does not create this branch and does not compute its source amplitude. GU III records the legality conditions under which the declared GU II source-class packet may be imported by GU IV, parent-source, or downstream GU layers.

Import Condition 7.1 (Sterile dense-fermion branch legality)**.** A sterile dense-fermion source-class packet is legal for downstream import only if the following gates are declared and satisfied on the relevant production interval:

(S1) Carrier gate. The occupied carrier is weak-singlet, sphaleron-blind, or otherwise protected over the production interval.

(S2) Anomaly gate. The carrier either belongs to the GU II anomaly-safe ledger or carries a separately declared anomaly-canceling completion.

(S3) Operator-basis gate. The branch couples to the same internal-singlet Lorentz axial current entering $O_{55} = -J_5^2$.

(S4) Washout gate. The chirality/spin-flip rate satisfies

$$\Gamma_\chi \tau_{\text{prod}} \lesssim 1.$$

(S5) Energy-budget gate. The dense sector satisfies

$$\rho_{\text{deg}} \lesssim \rho_b$$

on the declared parent branch.

(S6) Boundary and BRST gate. The source-class import is compatible with the declared BRST-preserving boundary family and the BV closure hypotheses used in GU III.

(S7) Sign-corridor gate. The downstream use of $\sigma_0^2 = C_{55}(\mu_{\text{eff}})n_{5,0}^2$ records either theorem-level cone invariance or a declared sign-corridor assumption for $C_{55}(\mu_{\text{eff}}) > 0$.

Scope Note 7.2 (Legality versus source production)**.** The sterile dense-fermion branch enters GU III as a legality import problem. The branch data

$$T_{\text{spin}}, \quad \bar{\mu}, \quad \mu_5, \quad P_A(r), \quad \Gamma_\chi \tau_{\text{prod}}, \quad \rho_{\text{deg}}/\rho_b$$

are GU II source-class data. GU III certifies import status. Parent-source values, tensor amplitudes, detector corridors, and observable residuals are outside GU III and enter GU IV only through declared source-adapter or observable-interface packets.

Export 7.3 (GU3-H sterile dense-fermion import export)**.** GU III exports the sterile dense-fermion import rule

$$\text{GU3-H}: \quad \text{GU II sterile dense-fermion source-class packet} \longrightarrow \text{legal downstream import}$$

only when theorem 7.1 is satisfied. Downstream use preserves the official branch name *sterile dense-fermion branch* and records the branch as source-class data, not as a GU III observable claim.

8 RG Matching and the Axial Sign Corridor

The matched axial coefficient is

$$\Delta L_X = C_{55}(\mu)O_{55}, \qquad C_{55}(\mu_{\mathrm{GU}}) > 0 \tag{8.1}$$

at the GU matching scale. The positive sign at matching is inherited from the GU I auxiliary torsion theorem. GU III specifies the conditions required for GU IV to keep using the positive sign after running and matching.

Definition 8.1 (Axial positive cone)**.** Let $C_i(\mu)$ be the coefficient vector for the closed four-fermion operator basis containing $O_{55} = O_{AA}$. The axial positive cone is

$$\mathcal{C}_{55}^{+} := \{C_i(\mu) : C_{55}(\mu) > 0 \text{ and the declared RG flow remains inside the cone}\}. \tag{8.2}$$

The perturbative running is written

$$\mu \frac{d}{d\mu} C_i(\mu) = \frac{1}{16\pi^2}\gamma_{ij}C_j(\mu). \tag{8.3}$$

The anomalous-dimension matrix γ_{ij} is scheme- and basis-dependent. GU III therefore exports a cone-invariance criterion, not a hidden numerical claim.

Theorem 8.2 (Axial sign-corridor cone criterion)**.** *Assume the operator basis is closed under* (8.3) *on the interval* $[\mu_{\mathrm{eff}}, \mu_{\mathrm{GU}}]$*, and assume the matrix* γ_{ij} *preserves the declared cone* $\mathcal{C}_{55}^{+}$*. Then*

$$C(\mu_{\mathrm{GU}}) \in \mathcal{C}_{55}^{+} \quad \Longrightarrow \quad C(\mu) \in \mathcal{C}_{55}^{+} \quad \textit{for all } \mu \in [\mu_{\mathrm{eff}}, \mu_{\mathrm{GU}}], \tag{8.4}$$

and in particular

$$C_{55}(\mu_{\mathrm{eff}}) > 0. \tag{8.5}$$

Proof. The proof is the standard invariance theorem for a finite-dimensional linear RG flow on a closed cone. If the vector field $\dot{C}_i = (16\pi^2)^{-1}\gamma_{ij}C_j$ is tangent to or points inward on the boundary of $\mathcal{C}_{55}^{+}$, then solutions beginning in the cone cannot cross its boundary. Therefore positivity of the O_{55} component at μ_{GU} remains valid throughout the interval. □

Sign-corridor status. In GU III, the axial sign corridor has one of two statuses:

1. **Theorem**, if the anomalous-dimension matrix is supplied and the positive cone is shown invariant by theorem 8.2.
2. **Declared sign-corridor assumption**, if the matrix is not supplied or if cone invariance is not verified.

GU IV may consume $C_{55}(\mu_{\text{eff}}) > 0$ only under one of these two statuses. The cone criterion is an import-consistency theorem: it states the condition under which the positive O_{55} sign propagates, but it does not itself compute the anomalous-dimension matrix or establish observational viability.

Diagnostic Note 8.3 (Positivity scope)**.** The sign $C_{55} > 0$ is inherited from the GU I auxiliary torsion matching, not from a generic forward-limit positivity theorem. Positivity diagnostics may test consistency of the axial channel, but they do not replace the matching theorem unless the amplitude, external states, helicities, subtraction scheme, IR treatment, pole removal, and operator dimension are specified.

8.1 Axial Sign Corridor and GU IV Handoff

The axial sign corridor is the final GU III bridge from quantum legality to the GU IV observable interface. The handoff object is the sign-safe contact

$$\Delta L_X = C_{55}(\mu)O_{55}, \qquad O_{55} = -J_{5\mu}J_5^{\mu}. \tag{8.6}$$

The sign statement is always a statement in the O_{55} basis. Equivalently, the coefficient of the raw mostly-plus contraction $J_{5\mu}J_5^{\mu}$ is $-C_{55}(\mu)$.

At the GU IV effective scale, the stiffness parameter is

$$\sigma_0^2 = C_{55}(\mu_{\text{eff}})n_{5,0}^2. \tag{8.7}$$

GU III certifies the legality of the sign and matching corridor. GU IV decides how this legally certified packet enters observables. The handoff (8.7) is valid only when the GU II current ledger supplies a conserved or effectively conserved axial branch and when the GU III sign-corridor status is either theorem-level or explicitly declared as a sign-corridor assumption.

9 Quantum Projection–Variation and Boundary BRST

Classical fixed-slice Projection–Variation is imported from GU I. GU III upgrades it to the gauge-fixed/BV functional under admissible boundary families.

Theorem 9.1 (Quantum Projection–Variation)**.** *Let $\Gamma_Y[\Phi]$ be the gauge-fixed effective functional on Y, and let $\Gamma_X[\iota^*\Phi]$ be the induced slice functional. For BRST-preserving gauge fixing and an admissible boundary family $\mathfrak{B}_{\mathrm{adm}}$,*

$$\delta\Gamma_X[\iota^*\Phi] - \iota^*(\delta\Gamma_Y[\Phi]) = s\mathcal{K}_X + d_{\partial X}\mathcal{B}_X, \tag{9.1}$$

with

$$\int_{\partial X} \mathcal{B}_X = 0 \tag{9.2}$$

for the declared boundary family. Therefore BRST-closed physical observables satisfy

$$\langle \delta\Gamma_X \rangle_{\mathrm{phys}} = \langle \iota^*(\delta\Gamma_Y) \rangle_{\mathrm{phys}}\,. \tag{9.3}$$

Proof. The proof is given in Section I. The fixed-embedding pullback identity is the GU I classical input. Gauge fixing changes the variational identity by a BRST-exact term. The admissible boundary family cancels the boundary flux, yielding (9.1)–(9.3). □

Table 6: BRST boundary-admissibility ledger.

Boundary family	**BRST preserved?**	**Quantum PV valid?**	**Notes**
Dirichlet completed fields	yes	yes	Fixed $\iota^*\widehat{A}$, compatible ghost restriction.
Neumann completed curvature	yes under flux condition	yes	Boundary ghost flux vanishes.
Mixed/Robin	conditional	conditional	Requires stated linear boundary operator commuting with BRST.
Parity-leak nonadmissible family	no	no	Diagnostic obstruction; not a GU III export family.

10 Handoff to GU IV and GU V

GU IV consumes the legal EFT basis, admissible boundary conditions, matching coefficient, and RG/sign-corridor status. GU V audits the GU IV packet. Legality of the import packet is distinct from observational viability; GU IV tests the declared observable branch, and GU V audits the declared test. The downstream map is

$$(\alpha, \gamma, \mathrm{RG\ data}, n_{5,0}) \longmapsto \sigma_0^2 = C_{55}(\mu_{\mathrm{eff}})n_{5,0}^2. \tag{10.1}$$

Export 10.1 (GU III final legality packet)**.** The final GU III export packet is

$$\begin{aligned}\text{GU3-}\textsc{Final} = \{&s^2 = 0,\ s\widehat{A} = -D_{\widehat{A}}c,\ s\widehat{F} = -[\widehat{F}, c],\ (S_{\mathrm{BV}}, S_{\mathrm{BV}}) = 0,\\ &H^{0,4}(s|d)_{\dim\le 6,\mathrm{even}},\ \mathcal{A}_{\mathrm{total}} = 0,\ C_{55}(\mu_{\mathrm{eff}}) > 0 \text{ status},\\ &\delta\Gamma_X - \iota^*\delta\Gamma_Y = s\mathcal{K}_X + d_{\partial X}\mathcal{B}_X,\ \sigma_0^2 = C_{55}(\mu_{\mathrm{eff}})n_{5,0}^2,\\ &\text{GU3-H sterile dense-fermion branch import-legality status}\}.\end{aligned} \tag{10.2}$$

GU IV may import these objects only under the conditions listed in Table 4.

11 Conclusion

GU III closes the quantum legality layer of the Geometric Unity series. GU I supplies the completed classical geometry; GU II supplies the matter and current ledger; GU III certifies the BRST/BV, anomaly, counterterm, RG/matching, positivity-diagnostic, sterile-branch import, and boundary Projection–Variation conditions under which those upstream exports can be handed to GU IV. The result is the quantum legality certificate for the GU observable packet.

The principal exports are fixed. The affine semidirect BRST complex is nilpotent. The completed variables transform covariantly. The BV master action satisfies the classical master equation under the declared closure hypotheses. The parity-even dim$\le$ 6 counterterm ledger is explicit. The GU II anomaly table is consumed without adding ghost, boundary, or measure anomalies. The axial sign corridor separates theorem-level RG protection from declared sign-corridor assumptions and positivity diagnostics. The sterile dense-fermion branch is imported only as GU II source-class data and becomes legal downstream only after its anomaly, washout, BRST/BV, boundary, operator-basis, energy-budget, and sign-corridor gates are recorded.

The handoff to GU IV is therefore clean as an import statement. A GU III-legal packet remains subject to GU IV observable-interface tests and GU V audit. GU IV may build observables from the certified packet

$$\{\widehat{A}, \widehat{F}, T_{\mathrm{aug}}, O_{55}, C_{55}(\mu_{\mathrm{eff}}), n_{5,0}, \sigma_0, \text{GU3-H}\}$$

only under the import conditions stated in Table 4. In particular,

$$\sigma_0^2 = C_{55}(\mu_{\mathrm{eff}})n_{5,0}^2$$

is legal only on a GU II conserved or effectively conserved axial-current branch and only after the GU III sign-corridor status has been recorded. Sterile dense-fermion source-class packets require the additional GU3-H import ledger. With those conditions in place, the quantum legality layer is closed and GU IV may proceed to the observable interface.

A Gradings, Ghost Numbers, and Sign Conventions

The BRST differential s has ghost number +1, form degree 0, and odd Grassmann parity. Products are graded by total degree

$$|\alpha|_{\mathrm{tot}} = \deg_{\mathrm{form}}(\alpha) + \mathrm{gh}(\alpha) \pmod 2. \tag{A.1}$$

The bracket is the graded bracket

$$[\alpha, \beta] = \alpha \wedge \beta - (-1)^{|\alpha|_{\mathrm{tot}}|\beta|_{\mathrm{tot}}} \beta \wedge \alpha, \tag{A.2}$$

with the Lie-algebra product understood.

Table 7: Gradings and parities used in GU III.

Field	**Form degree**	**Ghost number**	**Grassmann parity**	**Role**
A	1	0	even	H-connection.
B	1	0	even	affine compensator.
$\widehat{A}$	1	0	even	completed connection.
c	0	1	odd	H-ghost.
η	1	1	odd	affine N-ghost.
$\bar{c}$	0	-1	odd	H-antighost.
$\bar{\eta}_\mu$	1	-1	odd	affine antighost matching one-form shift.
b_c	0	0	even	H Nakanishi–Lautrup field.
$b_{\eta\mu}$	1	0	even	affine Nakanishi–Lautrup field.
Ψ	0	0	odd	GU II matter spinor.
λ	0	1	odd	Lorentz/spin ghost.
Φ^*	$D - \deg \Phi$	$-1 - \mathrm{gh}(\Phi)$	opposite to Φ	BV antifield.

Condition A.1 (Sign convention for connection covariance)**.** This paper uses

$$s\widehat{A} = -D_{\widehat{A}}c = -dc - [\widehat{A}, c]. \tag{A.3}$$

The derivative term is part of the connection transformation. Statements omitting $-dc$ are shorthand only after passing from the connection to its curvature or after explicitly declaring an adjoint-tensor convention.

B Semidirect Algebra and Nilpotency Proof

This appendix proves $s^2 = 0$ on the affine gauge pair, ghosts, and completed curvature.

Lemma B.1 (Ghost nilpotency)**.** *The ghost transformations*

$$sc = -\frac{1}{2}[c, c], \qquad s\eta = -[\eta, c] \tag{B.1}$$

satisfy

$$s^2 c = 0, \qquad s^2 \eta = 0. \tag{B.2}$$

Proof. For c, using that s is an odd graded derivation,

$$s^2 c = -\frac{1}{2} s[c, c] = -\frac{1}{2}([sc, c] - [c, sc]). \tag{B.3}$$

Substituting $sc = -\frac{1}{2}[c, c]$ gives the graded Jacobi identity in the form

$$[[c, c], c] = 0. \tag{B.4}$$

Therefore $s^2 c = 0$.

For η,

$$\begin{aligned} s^2 \eta &= -s[\eta, c] = -[s\eta, c] + [\eta, sc] \\ &= [[\eta, c], c] - \frac{1}{2}[\eta, [c, c]]. \end{aligned} \tag{B.5}$$

The graded Jacobi identity gives

$$[[\eta, c], c] = \frac{1}{2}[\eta, [c, c]], \tag{B.6}$$

so $s^2 \eta = 0$. □

Lemma B.2 (Nilpotency on A and B)**.** *The transformations* (2.1)–(2.2) *satisfy*

$$s^2 A = s^2 B = 0. \tag{B.7}$$

Proof. It is useful to package the homogeneous ghost and affine ghost into the semidirect ghost

$$\mathfrak{c} = (c, \eta)$$

and the affine gauge pair into the semidirect gauge object

$$\mathfrak{A} = (A, B).$$

The transformations (2.1)–(2.4) are precisely the component form of the Maurer–Cartan BRST

system
$$s\mathfrak{c} = -\frac{1}{2}[\mathfrak{c}, \mathfrak{c}]_{\ltimes}, \qquad s\mathfrak{A} = -d\mathfrak{c} - [\mathfrak{A}, \mathfrak{c}]_{\ltimes},$$
where the affine ideal is abelian and carries the adjoint H-module action. Applying s once more gives
$$s^2\mathfrak{A} = -d(s\mathfrak{c}) - [s\mathfrak{A}, \mathfrak{c}]_{\ltimes} + [\mathfrak{A}, s\mathfrak{c}]_{\ltimes}.$$
Substituting $s\mathfrak{c} = -\frac{1}{2}[\mathfrak{c}, \mathfrak{c}]_{\ltimes}$ and using the graded Jacobi identity gives
$$s^2\mathfrak{A} = 0.$$
Taking the two components of this identity gives $s^2A = 0$ and $s^2B = 0$. In components, the cancellation is the standard cancellation between the homogeneous connection part and the covariant affine-translation part; no independent affine curvature or additional ghost equation is required because the affine ideal is abelian. □

Theorem B.3 (Nilpotency and completed-variable covariance)**.** *The semidirect BRST differential satisfies*
$$s^2A = s^2B = s^2c = s^2\eta = 0, \tag{B.8}$$
and the completed variables satisfy
$$s\widehat{A} = -D_{\widehat{A}}c, \qquad s\widehat{F} = -[\widehat{F}, c], \qquad s^2\widehat{A} = s^2\widehat{F} = 0. \tag{B.9}$$

Proof. Nilpotency on c and η is theorem B.1. Nilpotency on A and B is theorem B.2. The completed-variable covariance is theorem 2.3. Since $s\widehat{A}$ is the standard connection BRST transformation, the standard curvature computation gives $s\widehat{F} = -[\widehat{F}, c]$. Applying s again and using $sc = -\frac{1}{2}[c, c]$ gives $s^2\widehat{A} = s^2\widehat{F} = 0$. □

Proposition B.4 (Completed torsion covariance under import)**.** *Under theorem 3.1, the completed torsion block satisfies*
$$sT_{\mathrm{aug}} = -[\lambda, T_{\mathrm{aug}}]_{\mathrm{Lor}}, \qquad s^2T_{\mathrm{aug}} = 0. \tag{B.10}$$

Proof. This is the BRST infinitesimal form of the GU I completed spin-cocycle theorem. The nilpotency follows from $s\lambda = -\frac{1}{2}[\lambda, \lambda]$ and the Lorentz-representation Jacobi identity. □

C Gauge Fixing and Ghost Action

This appendix displays the nonminimal sector, gauge-fixing fermion, and ghost/shift-ghost operators used by GU III.

Definition C.1 (Nonminimal sector)**.** The nonminimal fields are
$$(\bar{c}, b_c), \qquad (\bar{\eta}_\mu, b_{\eta\mu}), \tag{C.1}$$

with

$$s\bar{c} = b_c, \quad sb_c = 0, \qquad s\bar{\eta}_\mu = b_{\eta\mu}, \quad sb_{\eta\mu} = 0. \tag{C.2}$$

They form contractible BRST pairs and do not alter the physical cohomology.

Construction C.2 (Gauge-fixing fermion)**.** Choose

$$\mathcal{F}_A = \nabla^\mu A_\mu, \qquad \mathcal{F}_B^\mu = B^\mu. \tag{C.3}$$

The gauge-fixing fermion is

$$\Psi_{\mathrm{gf}} = \int_X \left[\langle \bar{c}, \nabla^\mu A_\mu \rangle + \langle \bar{\eta}_\mu, B^\mu \rangle + \frac{\xi_A}{2} \langle \bar{c}, b_c \rangle + \frac{\xi_B}{2} \langle \bar{\eta}_\mu, b_\eta^\mu \rangle \right]. \tag{C.4}$$

The gauge-fixed action is obtained by setting

$$\Phi^* = \frac{\delta \Psi_{\mathrm{gf}}}{\delta \Phi}. \tag{C.5}$$

Proposition C.3 (Ghost and shift-ghost operators)**.** *Linearizing around a background $(\bar{A}, \bar{B})$, the homogeneous ghost operator is*

$$\mathcal{M}_A c = -\nabla^\mu D_{\bar{A}\,\mu} c - \nabla^\mu \eta_\mu, \tag{C.6}$$

while the affine shift-gauge operator in the algebraic shift gauge is

$$\mathcal{M}_B{}^\mu{}_\nu \eta^\nu = -\eta^\mu - [\bar{B}^\mu, c]. \tag{C.7}$$

The shift-ghost sector is therefore nonpropagating in the algebraic shift gauge, as expected for an affine Stueckelberg-type shift. Derivative gauges may be used only after declaring the corresponding longitudinal parametrization of the affine one-form.

Proof. The operators are obtained by applying s to the gauge functions $\mathcal{F}_A$ and $\mathcal{F}_B^\mu$. Since $sA = -D_A c - \eta$,

$$s(\nabla^\mu A_\mu) = -\nabla^\mu D_{A\mu} c - \nabla^\mu \eta_\mu. \tag{C.8}$$

Since $sB^\mu = -[B^\mu, c] - \eta^\mu$, the affine shift-gauge operator follows. □

D BV Master Functional

Let Φ^A denote the minimal field set, including A, B, c, η, matter fields, and the completed spin variables when the spin import condition is imposed. Each field has an antifield Φ_A^* with

$$\mathrm{gh}(\Phi_A^*) = -1 - \mathrm{gh}(\Phi^A), \qquad \mathrm{par}(\Phi_A^*) = \mathrm{par}(\Phi^A) + 1 \pmod 2. \tag{D.1}$$

The BV antibracket is

$$(F,G) = \int \left(\frac{\delta_R F}{\delta \Phi^A} \frac{\delta_L G}{\delta \Phi^*_A} - \frac{\delta_R F}{\delta \Phi^*_A} \frac{\delta_L G}{\delta \Phi^A} \right). \tag{D.2}$$

Theorem D.1 (Minimal BV functional)**.** *For the closed off-shell semidirect BRST algebra, the minimal BV action*

$$S_{\mathrm{BV}} = S_{\mathrm{cl}}[\Phi] + \int \Phi^*_A s\Phi^A \tag{D.3}$$

satisfies

$$(S_{\mathrm{BV}}, S_{\mathrm{BV}}) = 0. \tag{D.4}$$

Proof. The antibracket with S_{BV} acts as

$$(S_{\mathrm{BV}}, \Phi^A) = s\Phi^A. \tag{D.5}$$

The classical action is built from BRST-covariant completed variables and the GU II anomaly-safe matter ledger, so $sS_{\mathrm{cl}} = 0$. The second antibracket operation reproduces $s^2\Phi^A$, which vanishes by theorem B.3 and by the spin import condition when that sector is included. Therefore $(S_{\mathrm{BV}}, S_{\mathrm{BV}}) = 0$. Since the algebra closes off shell, no antifield-quadratic term is needed in the included sector. □

Scope Note D.2 (Open-algebra boundary)**.** If a future extension adds a sector whose transformations close only on shell or only up to field-dependent structure functions, the minimal antifield-linear functional requires the standard higher-antifield BV extension. Such an extension is not part of the GU III export unless explicitly supplied.

E Local BRST Cohomology Ledger

The local cohomology statement is made on the four-dimensional slice and is restricted to the parity-even sector through mass dimension six. Operators are classified modulo BRST-exact terms, total derivatives, equations of motion, and local field redefinitions.

Table 8: Parity-even dim≤ 6 local cohomology/counterterm ledger.

Dim.	Representative	Status	Notes
0	$\sqrt{-g}$	physical if cosmological term retained	Background sector; GU IV decides observational treatment.
2	R with dimensionful coefficient conventions	physical	Gravity normalization inherited from GU I.

Dim.	Representative	Status	Notes
4	$\widehat{F}_{\mu\nu}\widehat{F}^{\mu\nu}$	physical	Completed curvature only; raw affine-shift variables excluded.
4	$\bar{\Psi}i\not{D}\Psi$	physical	GU II matter table.
4	$\nabla_\mu J^\mu$	total derivative	Boundary term under admissible families.
4	parity-odd densities	separate sector	Included only with explicit boundary/parity ledger.
6	$O_{55} = -(\bar{\Psi}\gamma_\mu\gamma^5\Psi)(\bar{\Psi}\gamma^\mu\gamma^5\Psi)$	physical	Axial contact.
6	$O_{VV}, O_{VA}, O_{LL}, O_{RR}, O_{LR}$	physical/mixing basis	Closed four-fermion basis after internal singlet projection.
6	derivative current terms	redundant or physical by ledger	Classified by EOM and integration by parts.
6	curvature-current terms	conditional	Included only if GU IV retains curvature-current sector.
6	bilinear graviton-gauge mixing $h \cdot a$	absent / redundant	GU I quadratic health and BRST covariance exclude irreducible terms.

Theorem E.1 (Dimensional cohomology ledger)**.** *Within the declared parity-even dim$\leq$ 6 slice EFT class,*

$$H^{0,4}(s|d)_{\mathrm{dim}\leq 6,\mathrm{even}} = \mathrm{span}\{\textit{physical representatives in Table 8}\} \oplus s(\cdots) \oplus d(\cdots) \oplus \mathrm{EOM}. \quad \text{(E.1)}$$

The axial contact is represented by O_{55}, not by an unlabelled raw J_5^2 contraction.

Proof. The completed curvature and completed torsion inputs transform covariantly by the BRST results above; Shiab/Hodge pairings and GU II matter bilinears therefore generate BRST-closed local densities. Exact, total-derivative, and equation-of-motion representatives are quotient directions in local BRST cohomology. The parity-even dim$\leq$ 6 physical representatives are precisely those listed in Table 8; the four-fermion sector is closed by declaring the basis containing O_{55} and its allowed mixing companions. □

F Anomaly Ledger Consumption

GU III imports the GU II matter anomaly table and checks additional quantum sectors.

Table 9: GU III anomaly sources and status.

Anomaly source	**Status**
GU II matter triangle anomalies	vanish by GU II one-family anomaly ledger.
Mixed gravitational-gauge anomalies	vanish by the same GU II all-left-handed anomaly convention.
Witten $SU(2)$ anomaly	absent by GU II doublet count.
Affine ghost sector	nonchiral BRST gauge-sector field; no new chiral triangle anomaly.
Boundary parity densities	exact or excluded by the admissible boundary family.
Measure anomaly	follows the same representation ledger; no extra measure anomaly under imported matter content.
Sterile dense-fermion branch	legal only as GU II source-class data with declared anomaly, washout, energy-budget, boundary, operator-basis, and sign-corridor status; no amplitude or observable activation is imported.

Theorem F.1 (Anomaly consumption theorem)**.** *Under the GU II matter import and the boundary conditions of Table 6, the total GU III anomaly ledger vanishes:*

$$\mathcal{A}_{\text{GU III}} = 0. \tag{F.1}$$

Proof. The matter contribution is zero by the GU II anomaly table. Gauge and affine ghosts are BRST gauge-fixing fields and do not form new chiral matter representations. The affine ghost is an adjoint module ghost for the shift sector, not a new chiral fermion. Boundary parity terms are admissible only when exact or cancelled by the boundary family. The functional measure uses the same anomaly-free representation data. Thus all terms in (6.1) vanish. □

Condition F.2 (Anomaly-consumption condition)**.** If GU III changes the matter content, chirality convention, gauge group, hypercharge embedding, or family table relative to GU II, then the GU II anomaly ledger is no longer imported and requires recomputation before any GU IV export is legal.

Condition F.3 (Sterile dense-fermion branch legality gate)**.** The sterile dense-fermion branch is imported from GU II as source-class data, not as an activated observable branch. GU III may certify the branch for downstream import only when all of the following entries are declared:

1. the occupied species list and weak-singlet/sterile carrier status;

2. anomaly compatibility in the GU II matter/current ledger;

3. the washout gate $\Gamma_\chi \tau_{\mathrm{prod}} \lesssim 1$;

4. the energy-budget gate $\rho_{\mathrm{deg}} \lesssim \rho_b$;

5. boundary admissibility for the current branch;

6. membership of the induced axial contact in the O_{55} operator basis;

7. the sign-corridor status of $C_{55}(\mu_{\mathrm{eff}})$.

This gate certifies legality of the source-class packet. It does not compute a parent-source value, a tensor amplitude, a detector corridor, or a GU IV observable map. A packet failing the washout gate or the energy-budget gate is outside the GU III import lane even if its formal axial-current algebra is well-defined.

G RG and Matching of the Axial Contact

The four-fermion coefficient vector is

$$C = (C_{AA}, C_{VV}, C_{VA}, C_{LL}, C_{RR}, C_{LR}, \ldots)^T, \qquad C_{AA} = C_{55}. \tag{G.1}$$

The general one-loop running is

$$\mu \frac{d}{d\mu} C = \frac{1}{16\pi^2} \Gamma_{4\mathrm{F}} C, \tag{G.2}$$

where

$$\Gamma_{4\mathrm{F}} = \begin{pmatrix} \gamma_{AA,AA} & \gamma_{AA,VV} & \gamma_{AA,VA} & \cdots \\ \gamma_{VV,AA} & \gamma_{VV,VV} & \gamma_{VV,VA} & \cdots \\ \gamma_{VA,AA} & \gamma_{VA,VV} & \gamma_{VA,VA} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \tag{G.3}$$

is supplied by the declared EFT scheme and operator basis.

Condition G.1 (RG input condition)**.** A numerical sign-corridor theorem requires the explicit matrix $\Gamma_{4\mathrm{F}}$ in the same operator basis, scheme, and internal-index convention used by GU IV. Without that matrix, GU III exports the sign corridor as a declared sign-corridor assumption, not as a computed RG theorem. The sign-corridor ledger certifies import consistency for the axial coefficient; it does not by itself establish observational viability.

Proposition G.2 (RG-to-σ_0 map)**.** *If either theorem 8.2 applies or the sign corridor is declared as a sign-corridor assumption, then GU IV may import*

$$\sigma_0^2 = C_{55}(\mu_{\mathrm{eff}}) n_{5,0}^2 \tag{G.4}$$

on a GU II conserved or effectively conserved axial branch. This is an import-legality statement. It does not assert that the resulting branch is observationally admissible.

Proof. The coefficient at μ_{eff} is legal and positive under either accepted sign-corridor status. GU II supplies the branch condition and current scaling. Multiplying the legal coefficient by $n_{5,0}^2$ gives the GU IV stiffness parameter. Observational admissibility depends on the GU IV branch-domain and data-facing tests, not on the GU III sign-corridor ledger alone. □

H Positivity and Causality Diagnostic

Forward-limit positivity statements are not used as the source of the axial sign in GU III. They are diagnostic checks.

Diagnostic Note H.1 (Forward-amplitude diagnostic requirements)**.** A positivity diagnostic for a four-fermion channel requires:

1. external states and helicities;
2. crossing prescription;
3. forward limit and subtraction scheme;
4. IR pole subtraction and regulator choice;
5. whether the constrained amplitude coefficient is order s, order s^2, dimension six, or an induced dimension-eight combination;
6. the map from the amplitude coefficient to the Wilson basis containing O_{55}.

If these data are absent, the result is not a theorem about C_{55}.

Proposition H.2 (Matching sign versus positivity diagnostic)**.** *The statement $C_{55}(\mu_{\text{GU}}) > 0$ is a matching result inherited from the GU I auxiliary torsion sector. Positivity diagnostics can test consistency of the axial channel but do not replace the matching statement, the RG cone-invariance condition, or the GU IV viability test.*

Proof. The GU I auxiliary torsion elimination fixes the sign in the O_{55} basis. Dispersion positivity constrains particular amplitude coefficients under additional analyticity, crossing, subtraction, and IR assumptions. Those assumptions are logically independent from the algebraic auxiliary-field matching. Therefore the matching sign and the positivity diagnostic are distinct statements. □

I Quantum Projection–Variation

Let Φ denote the full gauge-fixed field multiplet. Classical fixed-embedding Projection–Variation from GU I states that pullback and variation commute up to boundary terms for the slice density. Gauge fixing adds BRST-exact terms.

Definition I.1 (Admissible BRST boundary family)**.** A boundary family $\mathfrak{B}_{\mathrm{adm}}$ is BRST-admissible if:

1. the boundary conditions are stable under s;
2. the ghost and antighost fluxes vanish or are paired by a boundary counterterm;
3. the gauge-fixed presymplectic flux is BRST-exact plus an exact boundary derivative;
4. nonadmissible parity-leak terms are excluded.

Theorem I.2 (Gauge-fixed quantum Projection–Variation identity)**.** *For fixed embedding $\iota : X \hookrightarrow Y$, BRST-preserving gauge fixing, and boundary family $\mathfrak{B}_{\mathrm{adm}}$,*

$$\delta\Gamma_X[\iota^*\Phi] - \iota^*(\delta\Gamma_Y[\Phi]) = s\mathcal{K}_X + d_{\partial X}\mathcal{B}_X, \tag{I.1}$$

with $\int_{\partial X} \mathcal{B}_X = 0$. For BRST-closed observables the BRST-exact term decouples, yielding

$$\langle \delta\Gamma_X \rangle_{\mathrm{phys}} = \langle \iota^*(\delta\Gamma_Y) \rangle_{\mathrm{phys}} . \tag{I.2}$$

Proof. The GU I fixed-embedding theorem gives equality modulo the pulled-back presymplectic boundary term. The gauge-fixed action differs from the classical action by an s-exact term generated by the gauge-fixing fermion plus boundary pieces. BRST-admissibility cancels the boundary flux and keeps the family stable under s. Hence the difference has the form $s\mathcal{K}_X + d_{\partial X}\mathcal{B}_X$. Physical observables are represented by BRST cohomology classes, so the BRST-exact contribution vanishes in physical matrix elements. □

J GU III Artifact and Reproducibility Protocol

The GU III reproducibility protocol requires the following artifacts for a complete release. If an item is absent, it is described as planned or not part of the present release; it is not described as released:

1. `brst_nilpotency_check.tex` or notebook;
2. `bv_master_check.tex`;
3. `cohomology_ledger.csv`;
4. `operator_basis_dim6.yaml`;
5. `anomaly_import_manifest.yaml`;
6. `rg_matrix_O55.yaml` or `sign_corridor_assumption_manifest.yaml`;
7. `sign_corridor_check.ipynb`;
8. `quantum_pv_boundary_ledger.yaml`;

9. `sterile_dense_fermion_legality_manifest.yaml`;

10. `gu3_export_manifest.yaml`.

Each released artifact records filename, version, date, checksum, and proof label.

Protocol J.1 (Artifact release protocol)**.** A GU III artifact may be called released only if it appears in the export manifest with a checksum and a proof label. Otherwise the paper may state that the protocol requires the artifact, but it may not claim the artifact has been released. Branch-specific legality artifacts, including the sterile dense-fermion branch legality manifest, are import certificates rather than amplitude or observational claims. The manifest records the occupied carrier, washout gate, energy-budget gate, operator-basis gate, boundary status, and sign-corridor status.

K GU IV Export Checklist

Table 10: Objects GU IV may import from GU III.

Object	**Condition**	**Status**
BRST complex	$s^2 = 0$ proved for imported field basis.	required
Completed variables	$s\widehat{A} = -D_{\widehat{A}}c$, $s\widehat{F} = -[\widehat{F}, c]$.	required
Completed torsion	GU I spin-cocycle import condition imposed.	conditional
BV action	classical master equation checked.	required
Dim≤ 6 basis	local cohomology/counterterm ledger declared.	required
Anomaly closure	GU II ledger consumed; no new ghost/boundary/measure anomaly.	required
$C_{55}(\mu_{\mathrm{eff}}) > 0$	RG sign corridor theorem or declared sign-corridor assumption.	required
Quantum PV	boundary BRST admissibility.	required
Positivity	theorem or diagnostic label.	required
σ_0^2 import	legal only on GU II conserved/effectively conserved axial branch.	required
Sterile dense-fermion branch	GU II source-class packet declared; GU III anomaly, washout, energy-budget, boundary, operator-basis, and sign-corridor status recorded; no parent-source value, tensor amplitude, detector corridor, or observable map imported.	conditional

Review Criterion K.1 (Legality versus viability). A GU III-legal import packet is not automatically observationally admissible. GU III certifies BRST/BV, anomaly, counterterm, boundary, operator-basis, and sign-corridor status. GU IV supplies branch-domain and observable-interface tests, and GU V audits the declared packet.

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